1 Eigenvectors and eigenvalues

Eigenvectors of a matrix A represent directions x in which the action of the matrix by multiplication does not change the direction of x, but perhaps only the magnitude by a factor λ.

Definition 1. Let A be a square matrix of order n. A complex number λ is said to be **eigenvalue** for the matrix A if there exist a non-zero vector $x \in \mathbb{C}^n$ such that

$$Ax = \lambda x.$$

The vector x is called an **eigenvector** for A.

Example 2. Consider $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ and $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda = 3$. The multiplication Ax gives:

$$Ax = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda x.$$

Hence $\lambda = 3$ is an eigenvalue of A and the vector x is an eigenvector.

Theorem 3. *If A is a square matrix of order n, then λ is an eigenvalue of A if and only if it satisfies the equation*

$$\det(A - \lambda I_n) = 0.$$

*This is called the characteristic equation of A and $p_A = det(A - \lambda I_n)$, the **characteristic polynomial** of A.*

Example 4. For the matrix $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$, the **characteristic equation** is given by

$$A - \lambda I_2 = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = (\lambda - 3)(\lambda + 1) = 0,$$

and the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -1$.

Example 5. For the matrix $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$, the characteristic equation is given by

$$A - \lambda I_3 = \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = (\lambda - 2)^2(\lambda - 1) = 0,$$

and the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$.

For $\lambda_1 = 1$, the eigenvectors are $x = t\begin{pmatrix}3\\-1\\3\end{pmatrix}$, where $t \neq 0$.

For $\lambda_1 = 2$, the eigenvectors are solutions to the system:

$$\begin{cases} 3x_1 - 6x_2 - 6x_3 = 0 \\ -x_1 + 2x_2 + 2x_3 = 0 \\ 3x_1 - 6x_2 - 6x_3 = 0 \end{cases}$$

The solutions we get are $x = t\begin{pmatrix}2s+2t\\s\\r\end{pmatrix} = s\begin{pmatrix}2\\1\\0\end{pmatrix} + t\begin{pmatrix}2\\0\\1\end{pmatrix}$, where t, s can be any real numbers with the only condition that they cannot be both zero.

Example 6. The 90 degree rotation has matrix $Q = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$. After a rotation, no real vector Qx stays in the same direction as x, as long as $x \neq 0$. The eigenvalues of Q are the complex numbers $\lambda_1 = i$ and $\lambda_2 = -i$. The corresponding eigenvectors are given by:

$$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = -i\begin{pmatrix}1\\i\end{pmatrix} \qquad \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}i\\1\end{pmatrix} = i\begin{pmatrix}i\\1\end{pmatrix}.$$

In general a rotation by an angle θ has matrix $Q = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}$ without real eigenvalues.

Proposition 7. *(The 2×2 case) Consider the matrix $A = \begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}$. The eigenvalues of A are the roots of the equation:*

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

and therefore, in case both eigenvalues are real, we have the three possibilities:
(A) Both eigenvalues are positive $\Leftrightarrow$ $|A| > 0$ *and* $\mathrm{tr}(A) > 0$.
(B) Both eigenvalues are negative $\Leftrightarrow$ $|A| > 0$ *and* $\mathrm{tr}(A) < 0$.
(C) The two eigenvalues have different sign $\Leftrightarrow$ $|A| < 0$.

Theorem 8. *Let A be a square matrix of order n with characteristic polynomial*

$$p(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}b_{n-1}\lambda^{n-1} + \cdots + (-1)^1 b_1\lambda + b_0.$$

The coefficient b_k is given by the sum of all principal minors of A of order $n-k$. In particular:
(a) $b_0 = |A| = \det(A) = \lambda_1\lambda_2\ldots\lambda_n$.
(b) $b_{n-1} = \mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Theorem 9. *(Cayley-Hamilton) Any square matrix A satisfies its own characteristic polynomial. This means:*

$$p_A(A) = (-A)^n + b_{n-1}(-A)^{n-1} + \cdots + b_1(-A) + b_0 = 0.$$

In particular for $n = 2$, we get the identity:

$$A^2 = \operatorname{tr}(A)A - |A|.$$

Practice Questions:

1. Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$.